# Numerical approximation of the control for the wave equation

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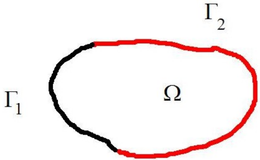
# Exact controllability of the Wave Equation

$$\begin{cases} \rho(x)u'' - \text{ div } (a(x)\nabla u) = 0 & \text{for } x \in \Omega, \quad t > 0 \\ u(t,x) = 0 & \text{for } t > 0, \text{ and } x \in \Gamma_1 \\ u(t,x) = v(t,x) & \text{for } t > 0, \text{ and } x \in \Gamma_2 \\ (u(0,x), u'(0,x)) = (u^0(x), u^1(x)) & \text{for } x \in \Omega \end{cases} \tag{1}$$

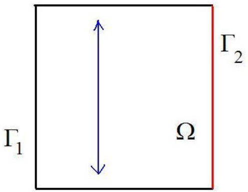
**Problem:** Given $T > T_0$ and $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ find a control function $v \in L^2(\Gamma_1 \times (0, T))$ such that

$$u(T, \cdot) = u'(T, \cdot) = 0. \tag{2}$$



# Necessary conditions:

1. Time $T_0$ must be sufficiently large
2. $\Gamma_2$ must satisfy a geometric condition: no trapping rays




3. $\rho(x)$ and $a(x)$ must be sufficiently smooth.

**Theorem** If the above conditions hold then the system is controllable

**Problem** Find a numerical approximation of the control

$$\mathcal{J} : H_0^1(0,1) \times L^2(0,1) \to R$$

$$\mathcal{J}(w_T^0, w_T^1) = \frac{1}{2}\int_0^T \int_{\Gamma_2} |\partial_n w|^2 + \int_\Omega u^0(x) w'(0,x)dx - \left\langle u^1, w(0,\cdot)\right\rangle_{-1,1}$$

$(w, w')$ being solution of the homogeneous backward equation

$$\left\{\begin{array}{ll} \rho(x)w'' - \text{ div } (a(x)\nabla w) = 0 & \text{for } x \in \Omega, \quad t > 0 \\ u(t,x) = 0 & \text{for } t > 0, \text{ and } x \in \partial\Omega \\ w(T,x) = w_T^0(x) & \text{for } x \in \Omega \\ w'(T,x) = w_T^1(x) & \text{for } x \in \Omega \end{array}\right. \tag{3}$$

Assume that $\mathcal{J}$ has a minimizer $(\widehat{w}_T^0, \widehat{w}_T^1)$. If $(\widehat{w}, \widehat{w}')$ is the corresponding solution of (3) with initial data $(\widehat{w}_T^0, \widehat{w}_T^1)$ then $v = \partial_n \, \widehat{w}_x|_{\Gamma_2}$ is the control of minimal $L^2-$norm (HUM control).

Theorem Assume that $E(0) \leq C \int_0^T \int_{\Gamma_2} (\partial_n w)^2$ then $\mathcal{J}$ has a minimizer. Moreover

$$\|\partial_n \widehat{w}_x\|_{L^2} \leq C\|(u^0, u^1)\|_{L^2 \times H^{-1}}$$

**Main question:** How to discretize the controlled system, the adjoint system and the functional $\mathcal{J}$ to obtain a convergent approximation of the control $v$.

# Example: The semi-discrete 1-D Wave Equation

$N \in \mathbb{N}^*$, $h = \frac{1}{N+1}$, $x_j = jh$, $0 \leq j \leq N+1$.

$$\left\{ \begin{array}{l} u_j''(t) = \frac{u_{j+1}(t)+u_{j-1}(t)-2u_j(t)}{h^2},\ t > 0 \\ u_0(t) = 0,\ t > 0 \\ u_{N+1}(t) = v_h(t),\ t > 0 \\ u_j(0) = u_j^0,\ \ u_j'(0) = u_j^1,\ 1 \leq j \leq N. \end{array} \right. \qquad (4)$$

**Discrete control problem:** Given $T > 0$ and $(U_h^0, U_h^1) = (u_j^0, u_j^1)_{1\leq j\leq N} \in \mathbb{R}^{2N}$, find a control function $v_h \in L^2(0, T)$ such that the solution $u$ of (4) satisfies

$$u_j(T) = u_j'(T) = 0,\ \ \forall j = 1, 2, ..., N. \qquad (5)$$

System (4) consists of $N$ linear differential equations with $N$ unknowns $u_1, u_2, ..., u_N$. $u_j(t)$ is an approximation of the solution $u$ in $(t, x_j)$, provided that $(U_h^0, U_h^1)$ approximates the initial datum $(u^0, u^1)$.

# Main questions

- Existence of the discrete control $v_h$.
- Boundedness of the sequence $(v_h)_{h>0}$ in $L^2(0, T)$.
- Convergence of the sequence $(v_h)_{h>0}$ to a control $v$ of the wave equation.

## Adjoint system:

$$\left\{\begin{array}{l} w_j''(t) = \frac{w_{j+1}(t)+w_{j-1}(t)-2w_j(t)}{h^2},\ t>0 \\ w_0(t) = w_{N+1}(t) = 0,\ t>0 \\ w_j(T) = w_j^0,\ \ w_j'(T) = w_j^1,\ 1\le j\le N \end{array}\right. \quad (6)$$

$$E_h(t) = \frac{h}{2}\sum_{j=0}^{N}\left[\left|w_j'(t)\right|^2 + \left|\frac{w_{j+1}(t)-w_j(t)}{h}\right|^2\right].$$

**Observability inequality:**

$$E_h(0) \le C\int_0^T \left(\frac{w_N}{h}(t)\right)^2 dt? \quad (7)$$

$$E_h(t) \approx \frac{1}{2}\int_0^1 \left((w'(t,x))^2 + (w_x(t,x))^2\right) dx$$

$$-\frac{w_N}{h}(t) \approx w_x(t,1).$$

Main difficulty: $C$ depends of $h$ and $\lim_{h\to 0} C(h) = \infty$. In fact,

$$\sup_u \frac{E_h(0)}{\int_0^T \left|\frac{w_n}{h}\right|} \geq c e^{1/h}$$

Consequence: There are initial data of the wave equation (even very regular ones) for which

$$\|v_h\|_{L^2} \to \infty \quad \text{as } h \to 0$$

**Conclusion:** A natural finite difference approximation for the wave equation do not provide a convergent algorithm!! (Glowinski R., Li C. H. and Lions J. L., 1990)

- **Filtering I.** $C$ is uniformly bounded if the initial data $(w_j^0, w_j^1)$ belong to the space generated by the first $\delta N$ eigenvectors with $\delta < 1$ (S. Micu, 2003).
- **Filtering II.** Uniform controllability of the projection of the solutions holds over the space generated by the first eigenvectors. The dimension of this space tends to infinity as the step size goes to zero (J.A. Infante and E. Zuazua, 1999; S. Ervedoza 2009 for the N-D).
- **Two grid algorithm.** Introduced by Glowinski and Li in 1993. Converge results by M. Negreanu and E. Zuazua 2004 and L. Ignat and E. Zuazua 2005.
- **Wavelets.** M. Negreanu, A.M. Matache and C. Schwab (2006)
- **Tychonov regularization.** Proposed by R. Glowinski, C.H. Li and J.-L. Lions in 1990. A uniform convergence result was obtained by L.R. Tcheugoue Tebou and E. Zuazua (2003)
- **Mixed finite element methods.** C.C. and S. Micu (2006)

# The variable coefficients wave equation

The solutions of the wave equation with an $L^\infty$ potential $a(x)$,

$$\left\{\begin{array}{ll} u_{tt} - u_{xx} + a(x)u = 0 & \text{for } x \in (0,1), \quad t > 0 \\ u(t,0) = u(t,1) = 0 & \text{for } t > 0, \\ u(0,x) = u^0(x) & \text{for } x \in (0,1) \\ u'(0,x) = u^1(x) & \text{for } x \in (0,1) \end{array}\right. \tag{8}$$

satisfy the following property (E. Zuazua-93):

$$E(0) \leq C_1 e^{C_2\sqrt{\|a(x)\|_{L^\infty}}} \int_0^T |u_x(0,t)|^2 dt,$$

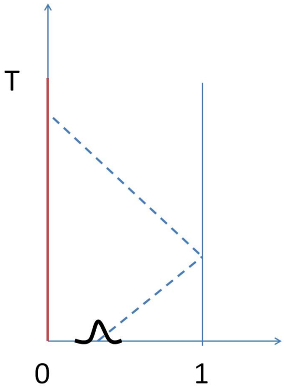
where $T > 2$ and

$$E(t) = \int_0^1 |u_t(x,t)|^2 dx + \int_0^1 |u_x(x,t)|^2 dx.$$

**Main question:** Is there a space semidiscretization of the wave equation that preserve this property?

# The case $a(x) = 0$

Observability is related with the speed of propagation. To observe at $x = 0$ we have to be aware of all disturbances induced by the initial data.



# Computing the velocity of propagation ($a(x) = 0$)

When considering solutions of the form

$$u(x,t) = exp(i\xi x - \omega(\xi)t), \quad \xi \in (-\pi/2, \pi/2),$$

we obtain the dispersion relation

$$\omega(\xi) = \pm|\xi|, \quad \xi \in (-\pi, \pi),$$

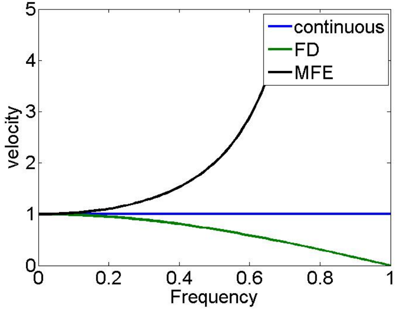
and the group velocity of waves

$$v(\xi) = \frac{d\omega}{d\xi} = \pm 1, \quad \xi \in (-\pi/2, \pi/2),$$

This explains why the time for the observability must be greater than 2.

# Understanding the case $a(x) = 0$

This group velocity can be also computed for discrete approximations

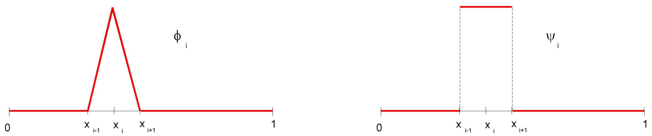



**Question:** Is this Mixed finite elements approach robust enough to deal with a potential?

**Main idea:** Large frequencies should be OK ...

Main idea:

$$u = \sum u_h^k \phi_k, \qquad u_t = \sum v_h^k \psi_k$$




See F. Brezzi and M. Fortin - 91

Matrix formulation:

$$\begin{cases} M_h U_h'' + K_h U_h + L_h U_h = 0, & t > 0, \\ U_h(0) = U_h^0, \quad U_h'(0) = U_h^1. \end{cases}$$

$$K_h = \frac{1}{h}\begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 \end{pmatrix}, \quad M_h = \frac{h}{4}\begin{pmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 2 \end{pmatrix},$$

$$L_h = h\begin{pmatrix} a_1 & 0 & 0 & \dots\dots & 0 \\ 0 & a_2 & 0 & \dots\dots & 0 \\ \dots & \dots & \dots & \dots\dots & \dots \\ 0 & 0 & 0 & \dots\dots & a_N \end{pmatrix}, \qquad U_h = \begin{pmatrix} U_{1,h} \\ U_{2,h} \\ \dots \\ U_{N,h} \end{pmatrix}.$$

# Main result: Uniform observability

**Theorem (C.C and S. Micu, 2019)**

*Assume $a_j \geq 0$ (equivalent to $a(x) \geq 0$). There exist constants $C, T > 0$, independent of $h$, such that*

$$E_h(U_h(0)) \leq C \int_0^T \left| \frac{U_{1,h(t)}}{h} \right|^2 dt$$

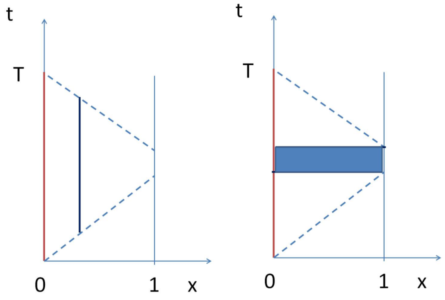
*where*

$$E_h(U_h) = (M_h U_h', U_h') + (K_h U_h, U_h)$$

Try to mimic the following proof for the continuous wave equation (Zuazua, 1993). Consider for $\tau > 2$ and $1 < \beta < \tau/2$

$$F(x) = \int_{\beta x}^{\tau - \beta x} \mathcal{E}(x,s)\, ds,$$

where $\mathcal{E}(x,t) = \frac{1}{2}(|u_t|^2 + |u_x|^2 + \|a\|_{L^\infty}|u|^2)$.



Analytically,

1. Consider for $\tau > 2$ and $1 < \beta < \tau/2$

$$F(x) = \int_{\beta x}^{\tau - \beta x} \mathcal{E}(x,s)\, ds,$$

   where $\mathcal{E}(x,t) = \frac{1}{2}(|u_t|^2 + |u_x|^2 + \|a\|_{L^\infty}|u|^2)$.
2. Prove $F'(x) \leq CF(x)$, for some constant $C > 0$.
3. By Gronwall's inequality $F(x) \leq cF(0)$
4. Using the conservation of the energy prove that

$$\mathcal{E}(0) \leq C_1 \int_0^1 F(x)\, dx \leq C_2 F(0).$$

At the discrete level, define

$$\mathcal{E}_j^h(s) = \left| \frac{U_{j+1} - U_j}{h} \right|^2 + \left| \frac{U'_{j+1} + U'_j}{2} \right|^2 + a_M \left| \frac{U_{j+1} + U_j}{2} \right|^2,$$

Consider also $\tau > 2$, $1 < \beta < \tau/2$ and the discrete version of $F(x)$:

$$F_j^h = \frac{1}{2} \int_{\beta x_j}^{\tau - \beta x_j} \mathcal{E}_j^h(s) ds.$$

## Lemma

*The following discrete version of $F'(x) \leq cF(x)$ holds*

$$\frac{F_j^h - F_{j-1}^h}{h} \leq c(a_M) \left( \frac{F_j^h + F_{j-1}^h}{2} + R_j^h(\beta x_j) + R_j^h(\tau - \beta x_j) \right),$$

$$R_j^h(s) = \frac{1}{h} \int_{s-\beta h}^{s} \mathcal{E}_j^h(r) dr - \frac{\mathcal{E}_j^h(s - \beta h) + \mathcal{E}_j^h(s)}{2},$$

The proof does not work!!

**Remark.** For particular solutions having only two frequencies $\lambda_n^h$, $\lambda_m^h$ with

$$|\lambda_n^h - \lambda_m^h| < 1,$$

we have

$$\frac{F_j^h - F_{j-1}^h}{h} \leq c(a_M)\frac{F_j^h + F_{j-1}^h}{2}$$

and the proof works!

The main ingredient is this lemma:

## Lemma

*Let $r > 0$, $t \geq 0$ and $\nu_1$, $\nu_2$ be two different real numbers such that,*

$$r|\nu_2 - \nu_1| \leq \frac{2\pi}{3}.$$

*Then, the following estimate holds*

$$\frac{f(t) + f(t+r)}{2} \leq \frac{5}{r} \int_t^{t+r} f(s)\, ds,$$

*for any function $f(t)$ of the form*

$$f(t) = \left| b_1 e^{i\nu_1 t} + b_2 e^{i\nu_2 t} \right|^2,$$

*with $b_1, b_2 \in \mathbb{C}$.*

The situation so far...

**Proposition**

*A uniform observability inequality holds but for particular solutions having only two frequencies $\lambda_n^h$, $\lambda_m^h$ with*

$$|\lambda_n^h - \lambda_m^h| < 1.$$

**Here we change the strategy of the proof.**

It is well known that the observability inequality can be obtained from two main properties:

1. A uniform spectral gap

$$\inf_{n \neq m} |\lambda_n^h - \lambda_m^h| > \gamma > 0,$$

where $\{\lambda_n^h\}_n$ are the frequencies.

2. A uniform observability inequality for the eigenfunctions.

- The uniform observability of the eigenfunctions is obtained from the discrete version of the continuous proof, that works fine for solutions having only one frequency.
- The espectral gap follows from the combination of the previous proposition and the following one:

### Proposition

*Assume that the uniform observability inequality holds for particular solutions having two frequencies $\lambda_h^n$, $\lambda_h^m$, then there exists a constant $C(T)$, uniform in $h$, such that*

$$|\lambda_n^h - \lambda_m^h| \geq C(T).$$

**Idea of the proof:**

$$\int_0^T \left| a_1 e^{i\lambda t} + a_2 e^{i\mu t} \right|^2 \geq C_1(|a_1|^2 + |a_2|^2) \Rightarrow |\lambda - \mu| > C_2(T, C_1)$$

The situation is as follows: For solutions having only two frequencies we have

$$|\lambda_n^h - \lambda_m^h| < 1 \Rightarrow \text{ Uniform observability}$$

$$\text{Uniform observability} \Rightarrow |\lambda_n^h - \lambda_m^h| > C(T)$$

Therefore,

$$\inf_{n \neq m} |\lambda_n^h - \lambda_m^h| > \min(1, C(T)).$$

and the spectral gap condition holds.

# Some comments

1. The only condition for the discrete potential $a_j^h$ is $0 < a_j^h < a_M$ for all $j = 1, ..., N$.
2. The proof can be adapted to non-positive potentials. This case is more technical.
3. The time for observability is larger than the continuous one and probably not optimal.
4. The proof cannot be adapted to higher dimensions.

Consider the wave equation with a boundary control

$$\begin{cases} u_{tt} - u_{xx} + a(x)u = 0, & x \in (0,1), \quad t \in (0,T) \\ u(0,t) = 0, \qquad u(1,t) = f(t), & t \in (0,T) \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x) & \end{cases}$$

After,

1. homogenization of the problem
2. Fourier representation of the data and solution
3. truncate the series

we obtain the following system for the Fourier coefficients $y = (y_1, ..., y_n)^t$

$$\begin{cases} \frac{d^2y}{dt^2} = Ay(t) + Bu(t), & t \in [0,T] \\ y(0) = y^0, \qquad y'(0) = y^1 & \end{cases}$$

Let $\gamma$ be the solution of

$$\left\{\begin{array}{ll} \gamma''(x) - a(x)\gamma(x) = 0, & x \in (0,1), \\ \gamma(0) = 0, \quad \gamma(1) = 1. & \end{array}\right.$$

Then, the change of variables $v = u - \gamma(x)f(t)$ transform

$$\left\{\begin{array}{lll} u_{tt} - u_{xx} + a(x)u = 0, & x \in (0,1), & t \in (0,T) \\ u(0,t) = 0, \qquad u(1,t) = f(t), & t \in (0,T) & \\ u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x) & & \end{array}\right.$$

into

$$\left\{\begin{array}{lll} v_{tt} - v_{xx} + a(x)v = -\gamma(x)f''(t), & x \in (0,1), & t \in (0,T) \\ v(0,t) = 0, \qquad v(1,t) = 0, & t \in (0,T) & \\ v(x,0) = u^0(x), \quad v_t(x,0) = u^1(x) & & \end{array}\right.$$

# Fourier representation of the solution

Write

$$v = \sum_{j=1}^{\infty} v_j(t) \sin(j\pi x)$$

and the initial conditions

$$u^0(x) = \sum_{j=1}^{\infty} u_j^0 \sin(j\pi x),$$

$$u^1(x) = \sum_{j=1}^{\infty} u_j^1 \sin(j\pi x),$$

We obtain

$$\sum_{j=1}^{\infty} (v_j''(t) + j^2\pi^2 v_j + a(x)v_j + \gamma(x)f''(t)) \sin(j\pi x) = 0$$

# Truncation and projection

For $k = 1, ..., N$,

$$\begin{cases} v_k''(t) + k^2\pi^2 v_k + \sum_{j=1}^N v_j \int_0^1 a(x)\sin(j\pi x)\sin(k\pi x)dx \\ \quad = -f''(t) \int_0^1 \gamma(x)\sin(k\pi x)dx \\ v_k(0) = v_k^0, \\ v_k'(0) = v_k^1. \end{cases}$$

This is a system of ordinary differential equations for which the control with minimal $L^2-$norm can be constructed explicitely.

**Theorem**

*Assume that $a(x) \in L^\infty$ and $a(x) \geq 0$. Then, $f_N(t) \to f(t)$ in $L^2(0, T)$ as $N \to \infty$, where $f$ is the HUM control of the continuous system.*

**Idea of the method:** Drive first Fourier coefficients to zero. The energy associated to the rest fourier coefficients remains small and this provides an approximation of the control.