Quasimodes and Control of a Quantum Particle

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Trends on PDE's and Related Fields

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Three problems and a geometric condition

I'll start by describing three well known problems arising in (linear) PDE and how they are related to each other.

On a closed and compact Riemannian manifold M, and for some open set $\omega \subset M$, we consider:

- Exponential decay for solutions to the **damped wave equation** (with damping supported on ω).
- Exact controlability of the **Schrödinger equation** (with controls supported on ω).
- Observability of **eigenfunctions of the Laplacian** (from the open set ω).

All these problems involve the **Laplace-Beltrami** operator (or simply the Laplacian) Δ_x on M.

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Sabilisation for the damped wave equation

This is the **damped wave equation**:

$$\begin{cases} \partial_t^2 u(t,x) - \Delta_x u(t,x) + a(x)\partial_t u(t,x) = 0, \quad (t,x) \in \mathbb{R} \times M, \\ (u(0,\cdot), \partial_t u(0,\cdot)) = (u^0, u^1) \in H^1(M) \times L^2(M). \end{cases}$$

Here, the **damping coefficient** is supposed to be non-negative; for instance:

$$a := c\mathbb{1}_\omega, \qquad \text{for some open set } \omega \subset M.$$

It is easy to check that the **energy**

$$\mathcal{E}(t,u) := \int_M (|\partial_t u(t,x)|^2 + |\nabla_x u(t,x)|^2) dx$$

of any solution u decays:

$$\frac{d}{dt}\mathcal{E}(t,u) = -2\int_M a(x)|\partial_t u(t,x)|^2 dx < 0.$$

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Stabilisation for the damped wave equation

A natural question to ask is whether or not this decay is uniform with respect to the initial data:

$$\mathcal{E}(t,u) \leq f(t)\mathcal{E}(0,u), \qquad \text{for some } f(t) \to 0^+ \text{ as } t \to +\infty,$$

for every initial data $(u^0, u^1) \in H^1(M) \times L^2(M)$. If such an $f(t)$ exists, one can check that is must be of the form:

$$f(t) = Me^{-\alpha t}, \quad \alpha, M > 0.$$

Hence the terminology **exponential decay** or stabilisation.

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Hence the terminology **exponential decay** or stabilisation.

Exponential decay is equivalent to observability

Another functional analytic argument gives:

Stabilisation is equivalent to observability

Uniform decay for the damped wave equation holds if and only if the **observability estimate**

$$\mathcal{E}(0,v) \leq C_{T,\omega} \int_0^T \int_\omega |\partial_t v(t,x)|^2 dx dt, \qquad (O_{\mathrm{W}}(\omega))$$

holds for some $T, C_{T,\omega} > 0$ uniformly for every solution of the **free wave equation**:

$$\begin{cases} \partial_t^2 v(t,x) - \Delta_x v(t,x) = 0, & (t,x) \in \mathbb{R} \times M, \\ (v(0,\cdot), \partial_t v(0,\cdot)) = (v^0, v^1) \in H^1(M) \times L^2(M). \end{cases}$$

Please, keep this estimate in mind, it will appear several times in the talk!

Controllability of the Schrödinger equation

We now turn to the **Schrödinger equation**:

$$\begin{cases} i\partial_t \Psi(t,x) + \Delta_x \Psi(t,x) = F(t,x), \\ \Psi(0,\cdot) = \Psi^0, \end{cases}$$

where the the forcing term satisfies:

$$F \in L^2_{\mathrm{loc}}(\mathbb{R} \times M), \quad F(t,x) = 0 \text{ if } x \in M \setminus \omega.$$

for some open set $\omega \subset M$.

The Schrödinger equation is **exactly (null) controllable** from ω at some time $T > 0$ provided that given any $\Psi^0 \in L^2(M)$, is it always possible to find a forcing term F as above such that the corresponding solution Ψ satisfies

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Controllability is equivalent to Observability

Again, a functional analysis argument shows that controllability from ω at time T is equivalent to an estimate.

Controllability is equivalent to Observability

The Schrödinger equation is controllable from ω at time T if and only if there exist a constant $C = C_{T,\omega} > 0$ such that every $\psi^0 \in L^2(\Omega)$ satisfies the **observability estimate**

$$||\psi^0||^2_{L^2(M} \leq C \int_0^T \int_\omega |\psi(t,x)|^2 dx\, dt, \qquad (O_S(\omega))$$

where ψ is the solution of the **homogeneous Schrödinger equation**

$$\begin{cases} i\partial_t \psi\,(t,x) + \Delta_x \psi\,(t,x) = 0, \\ \psi(0,\cdot) = \psi^0. \end{cases}$$

Notice that this estimate es essentially the same as the previous one, but the equation is different.

Eigenfunctions of the Laplacian

Since M is compact, the Laplacian can be diagonalized and its spectrum is discrete.

There exists an orthonormal basis (φ_k) of $L^2(M)$ consisting of **eigenfunctions of the Laplacian**:

$$-\Delta_x \varphi_k(x) = \lambda_k \varphi_k(x), \quad x \in M,$$

with

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \to +\infty.$$

Observability of eigenfunctions

An important question in Mathematical Physics is that of understanding the structure of high frequency eigenfunctions.

Observability of eigenfunctions

We say that eigenfunctions of the Laplacian are observable from ω if a constant $C = C_\omega > 0$ exits such that

$$||\varphi||_{L^2(M} \leq C_\omega \|\varphi\|_{L^2(\omega)}, \qquad (O_{\mathrm{E}}(\omega))$$

for every eigenfunction of the Laplacian:

$$-\Delta_x \varphi(x) = \lambda \varphi(x), \quad x \in M.$$

Note that the constant C_ω is required to be uniform with respect to λ!

$(O_{\mathrm{W}}(\omega))$ and $(O_{\mathrm{S}}(\omega))$ imply $(O_{\mathrm{E}}(\omega))$

If φ is an eigenfunction of the Laplacian of norm one and eigenvalue λ then:

$$v(t,\cdot) := e^{-it\sqrt{\lambda}}\varphi \text{ is a solution to the Wave Equation and } \mathcal{E}(0,v) = \lambda;$$

$$\psi(t,\cdot) = e^{-it\lambda}\varphi \text{ is a solution to the Schrödinger Equation.}$$

Note that:

$$|\partial_t v(t,\cdot)|^2 = \lambda|\varphi|^2, \quad |\psi(t,\cdot)|^2 = |\varphi|^2;$$

therefore

$$(O_{\mathrm{W}}(\omega)) \text{ or } (O_{\mathrm{S}}(\omega)) \implies (O_{\mathrm{E}}(\omega)).$$

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A necessary and sufficient condition for exponential decay of damped waves

The Geometric Control Condition

The open subset $\omega \subseteq M$ satisfies the **Geometric Control Condition (GCC)** provided that every geodesic of M intersects ω.

The GCC is **sufficient and (almost) necessary** for the uniform decay of damped waves.

Theorem (Rauch-Taylor '74, Bardos-Lebeau-Rauch '87)

- *Suppose that ω satisfies the GCC. Then $(O_W(\omega))$ holds.*
- *Suppose that $M \setminus \overline{\omega}$ contains a geodesic. Then $(O_W(\omega))$ does not hold.*

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A sufficient condition for controllability of the Schrödinger equation

The GCC is **sufficient** for the controllability of the Schrödinger equation.

Theorem (Lebeau '92)

Suppose that ω satisfies the GCC. Then $(O_{\mathrm{S}}(\omega))$ holds.

As a consequence:

$$(O_{\mathrm{W}}(\omega)) \implies (O_{\mathrm{S}}(\omega)).$$

However, GCC **is not necessary in general**:

Theorem (Jaffard '90, Anantharaman-M. '14, Bourgain-Burq-Zworski'14, Anantharaman-Fermanian-M. '15)

Suppose that M is a flat torus. Then $(O_{\mathrm{S}}(\omega))$ holds for every open set ω. And therefore, so does $(O_{\mathrm{E}}(\omega))$.

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An intermidiate case: the Euclidean disk

Suppose now that $M = \mathbb{D} = \{z \in \mathbb{R}^2 \,:\, |z| \leq 1\}$. Estimates $(O_{\mathrm{S}}(\omega))$ and $(O_{\mathrm{E}}(\omega))$ do not hold for every open set ω.

Theorem (Anantharaman-Léautaud-M. '16)

Estimates $(O_{\mathrm{S}}(\omega))$ and $(O_{\mathrm{E}}(\omega))$ hold if and only if $\omega \subseteq \mathbb{D}$ intersects the boundary $\partial\mathbb{D}$ in an open non-empty interval.

This new geometric condition is strictly weaker than the GCC.

The disk and the torus are geometries in which:

- $(O_{\mathrm{S}}(\omega))$ and $(O_{\mathrm{E}}(\omega))$ hold under the same condition on ω.
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Sometimes, the three estimates are equivalent

Suppose now $M = \mathbb{S}^2$ is the round sphere. In this case GCC is necessary for $(O_{\rm E}(\omega))$ (and therefore for $(O_{\rm S}(\omega))$.

Theorem

Suppose that $\mathbb{S}^2 \setminus \overline{\omega}$ contains a geodesic. Then $(O_{\rm E}(\omega))$ does not hold.

Therefore, in the sphere the three estimates $(O_{\rm W}(\omega))$,$(O_{\rm S}(\omega))$,$(O_{\rm E}(\omega))$ hold under exactly the same conditions on ω.

Proof

Write the sphere as:

$$\mathbb{S}^2 := \{x = (x_1, x_2, x_3) : |x_1|^2 + |x_2|^2 + |x_3|^2 = 1\}.$$

Let

$$\varphi_k(x) = c_k(x_1 + ix_2)^k, \quad \text{with } c_k \text{ such that } ||\psi_k^0||_{L^2(\mathbb{S}^2)} = 1.$$

This function is a spherical harmonic (eigenfunction of the Laplacian):

$$-\Delta\varphi_k(x) = k(k+1)\varphi_k(x), \quad x \in \mathbb{S}^2.$$

Clearly

$$|\varphi_k(x)|^2 = (c_k)^2(|x_1|^2 + |x_2|^2)^k = (c_k)^2(1 - |x_3|^2)^k.$$

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Proof (cont.)

If $\overline{\omega} \cap \{x_3 = 0\} = \emptyset$ then

$$||\varphi_k||_{L^2(\mathbb{S}^2} \leq C_\omega \|\varphi_k\|_{L^2(\omega)}, \qquad (O_{\mathrm{E}}(\omega))$$

fails, since:

$$\lim_{k\to\infty} \int_\omega |\varphi_k(x)|^2 dx = 0, \quad \text{and} \quad ||\varphi_k||_{L^2(\mathbb{S}^2)} = 1.$$

Since any other geodesic of $\mathbb{S}^2$ can be obtained by applying a rotation to $\{x_3 = 0\}$, and the composition of a spherical harmonic with a rotation is again a spherical harmonic, the claim follows.

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The remaining situation

Our main result is the existence of a geometric situation in which:

- $(O_{\mathrm{W}}(\omega))$ and $(O_{\mathrm{S}}(\omega))$ hold under the same condition on ω.
- $(O_{\mathrm{E}}(\omega))$ holds under a strictly weaker condition on ω.

Theorem (M.-Rivière '16)

There exist infinitely many surfaces of revolution M with the following properties:

- $(O_{\mathrm{W}}(\omega))$ *and* $(O_{\mathrm{S}}(\omega))$ *hold if and only if* ω *satisifies GCC.*
- M *has a finite family of geodesics* $\gamma_1, \cdots, \gamma_N$ *such that* $(O_{\mathrm{E}}(\omega))$ *holds as soon as:*

$$\gamma_i \cap \omega \neq \emptyset, \quad i = 1, \cdots, N.$$

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A simpler problem

Some aspects of the proof are a bit technical, but the main ideas are contained in the proof of an slightly simpler result on the sphere $M = \mathbb{S}^2$.
The idea is to replace $-\Delta_x$ by $-\Delta_x + V$ where $V \in \mathcal{C}^\infty(\mathbb{S}^2; \mathbb{R})$ is real-valued. It is not hard to prove that the addition of this potential does not affect the validity of $(O_{\mathrm{S}}(\omega))$:

Theorem (M. '09)

If $\gamma \cap \overline{\omega} = \emptyset$ for some geodesic γ (i.e. GCC fails) then

$$\|\psi^0\|^2_{L^2(\mathbb{S}^2)} \leq C \int_0^T \int_\omega |e^{it(\Delta_x - V)}\psi^0(x)|^2 dx\, dt,$$

fails for every $T > 0$.

But $(O_{\mathrm{S}}(\omega))$ may hold for almost every ω

Theorem (M.-Rivière '17)

There exists $V_{\mathrm{M}} \in \mathcal{C}^\infty(\mathbb{S}^2)$ with the following property. There exist three distinct geodesics $\gamma_1, \gamma_2, \gamma_3$ such that, for every ω with

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the estimate

$$\|\varphi\|_{L^2(\mathbb{S}^2)} \leq C\|\varphi\|_{L^2(\omega)}$$

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For this choice of ω, the estimate for the Schrödinger equation $(O_{\mathrm{S}}(\omega))$ may fail dramatically.

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For this choice of ω, the estimate for the Schrödinger equation $(O_{\mathrm{S}}(\omega))$ may fail dramatically.

This is a consequence of a more general principle

Consider the **Radon transform** of V:

$$\mathcal{I}(V)(x,\xi) = \frac{1}{2\pi}\int_0^{2\pi} V(\gamma_{x,\xi}(s))ds,$$

where $\gamma_{x,\xi}$ is the geodesic of $\mathbb{S}^2$ starting at $x \in \mathbb{S}^2$ with velocity $\xi \in T_x\mathbb{S}^2$, $\|\xi\|_x = 1$.

Theorem (M.-Rivière '17)

Suppose that ω interesects all the geodesics γ such that $d\mathcal{I}(V)(\gamma,\dot\gamma) = 0$ (critical points of the Radon transform of V).
Then

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Some remarks

- For genereric **even** V (meaning $V(x) = V(-x)$, the Radon transform $\mathcal{I}(V)$ has only a finite number of geodesics as critical points. Therefore, the condition on ω is simply that it has to intersect all the geodesics in this finite family.
- If V is **odd** (meaning $V(x) = -V(-x)$) then $\mathcal{I}(V) = 0$. However, we can prove a similar result replacing the Radon transform by a different nonlinear transform of V, whose expression is a bit more complicated.
- This result holds for spheres of any dimension, not only the two-dimensional sphere.
- The surface of revolution we construct is such that its Laplacian can be unitarily conjugated to an operator that is very similar to the Schrödinger operator $-\Delta + V$. Using an explicit computation due to Zelditch we are able to show that the analog of $\mathcal{I}(V)$ is not constant. Hence the result.

Construction of V_{M}

Identify the space of oriented geodesics on $\mathbb{S}^2$ with $\mathbb{S}^2$ as follows: if γ is a geodesic then it contained in a plane. The two unit normal vectors of this plane are identified to the two orientations of the geodesic. For instance:

$$\gamma = \{x_3 = 0\} \quad \text{is identified to} \quad (0,0,1) \text{ and} (0,0,-1).$$

With this identification in mind, the Radon transform is bijective as an operator:

$$\mathcal{I} : C^\infty_{\mathrm{even}}(\mathbb{S}^2) \longrightarrow C^\infty_{\mathrm{even}}(\mathbb{S}^2).$$

Define:

$$V_{\mathrm{M}} := \mathcal{I}^{-1}(W), \quad W(x) := ax_1^2 + bx_2^2 + cx_3^2, \quad a < b < c.$$

Then W has exactly six distinct critical points that correspond to three distinct geodesics. Note than an even function on the sphere cannot have less than six critical points.

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Construction of V_{M}

Identify the space of oriented geodesics on $\mathbb{S}^2$ with $\mathbb{S}^2$ as follows: if γ is a geodesic then it contained in a plane. The two unit normal vectors of this plane are identified to the two orientations of the geodesic. For instance:

$$\gamma = \{x_3 = 0\} \quad \text{is identified to} \quad (0,0,1) \text{ and} (0,0,-1).$$

With this identification in mind, the Radon transform is bijective as an operator:

$$\mathcal{I} : \mathcal{C}^{\infty}_{\text{even}}(\mathbb{S}^2) \longrightarrow \mathcal{C}^{\infty}_{\text{even}}(\mathbb{S}^2).$$

Define:

$$V_{\mathrm{M}} := \mathcal{I}^{-1}(W), \quad W(x) := ax_1^2 + bx_2^2 + cx_3^2, \quad a < b < c.$$

Then W has exactly six distinct critical points that correspond to three distinct geodesics. Note than an even function on the sphere cannot have less than six critical points.

Ideas of the proof

The proof is based on an argument by contradiction involving semiclassical defect measures.

It turns out that semiclassical defect measures associated to sequences of eigenfunctions of $-\Delta_x + V$ are probability measures concentrated on $S^*\mathbb{S}^2$ and are invariant by two flows: the geodeisc flow and the Hamiltonian flow associated to $\mathcal{I}(V)$.

Since these two flow commute, this forces any semiclassical measure to be supporte on the equilibrium points of the Hamiltonian vector field associated to $\mathcal{I}(V)$.

These are precisely, the critical points of $\mathcal{I}(V)$.

If the observation region ω intersects all the critical points and the estimate fails, we get a contradiction. The semiclassical measure of a sequence of eigenfunctions violating the estimate must vanish identically!

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