On the stabilization of a rotating disk-beam system with a finite memory term

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Outline

Motivation

- An elastic beam is attached to a rigid body which is assumed to rotate around its axis and the motion of the beam is confined to a plane perpendicular to the disk.

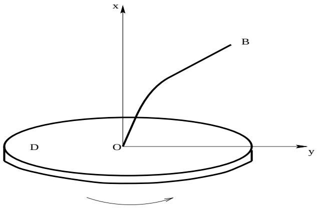

Fig. 1: The disk-beam system.

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- We assume that there is no distributed damping and the feedback law proposed consists of a linear control torque applied on the rigid body and a memory type boundary control force exerted at the free end of the beam.
- The aim is to stabilize the system by means of linear controls where a a finite memory term occurs.

Model

The system is governed by

$$\begin{cases} \rho(x)y_{tt} + (EI(x)y_{xx})_{xx} = \rho(x)\omega^2(t)y, & x \in (0,\ell), t > 0, \\ y(0,t) = y_x(0,t) = y_{xx}(\ell,t) = 0, & t > 0, \\ (EI(x)y_{xx})_x(\ell,t) = \mathcal{F}(t), & t > 0, \\ \dfrac{d}{dt}\left\{\omega(t)\left(I_d + \displaystyle\int_0^\ell \rho(x)y^2(x,t)dx\right)\right\} = \mathcal{T}(t), & t > 0, \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x), & x \in (0,\ell), \\ \omega(0) = \omega_0 \in \mathbb{R}. \end{cases} \tag{1}$$

Model

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- $EI(x)$, $\rho(x)$ and I_d are respectively the flexural rigidity, the mass per unit length of the beam, and the disk's moment of inertia.
- Moreover, $y(x,t)$ represents the beam's displacement at time t with respect to the spatial variable x, whereas ω is the angular velocity of the disk.
- Finally, $\mathcal{F}(t)$ and $\mathcal{T}(t)$ respectively involve the force control exerted on the free end of the beam and the torque control to be applied on the disk through which the system (1) is stabilized.

  Specifically, the aim is to suppress the beam vibrations and leave the disk rotating with a desired angular velocity ϖ.

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\begin{cases}
\mathcal{F}(t) = c^\top r(t) + \alpha y_t(\ell,t) + \beta \int_{t-\tau_2}^{t-\tau_1} \lambda(t-s) y_t(\ell,s)\, ds, \\
\dot{r}(t) = Ar(t) + by_t(\ell,t), \\
\mathcal{T}(t) = -\gamma(\omega(t) - \varpi), \quad \varpi \in \mathbb{R},
\end{cases}
\tag{2}
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in which $\alpha > 0$, $\beta \in \mathbb{R}$ and $\gamma > 0$ are feedback gains.

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in which $\alpha > 0$, $\beta \in \mathbb{R}$ and $\gamma > 0$ are feedback gains. $r \in \mathbb{R}^n$ is the actuator vector state, A is an $n \times n$ constant matrix and b, $c \in \mathbb{R}^n$ are constant vectors. Finally, $\tau_1, \tau_2 \in \mathbb{R}$ so that $0 \leq \tau_1 < \tau_2$ and $\lambda \in L^\infty(\tau_1, \tau_2)$ is the memory kernel.

Model

Letting $u(\zeta, t, s) = y_t(\ell, t - \zeta s)$, $\zeta \in (0,1)$, $s \in (\tau_1, \tau_2)$, the closed-loop system is then brought to the form

$$
\begin{cases}
\rho(x)y_{tt} + (EI(x)y_{xx})_{xx} = \rho(x)\omega^2(t)y, \\
y(0,t) = y_x(0,t) = y_{xx}(\ell,t) = 0, \\
(EI(x)y_{xx})_x(\ell,t) = c^\top r(t) + \alpha u(0,t,s) + \beta \displaystyle\int_{\tau_1}^{\tau_2} \lambda(s) u(1,t,s)\, ds, \\
\dot{r}(t) = Ar(t) + bu(0,t,s), \\
su_t(\zeta,t,s) + u_\zeta(\zeta,t,s) = 0, \qquad (\zeta,s) \in (0,1) \times (\tau_1,\tau_2), \\
\dfrac{d}{dt}\left\{\omega(t)\left(I_d + \displaystyle\int_0^\ell \rho(x)y^2(x,t)dx\right)\right\} = -\gamma(\omega(t) - \varpi), \\
y(x,0) = y_0(x),\ y_t(x,0) = y_1(x),\ \ \omega(0) = \omega_0 \in \mathbb{R}, \\
u(\zeta,0,s) = f_0(-\zeta s), \qquad (\zeta,s) \in (0,1) \times (0,\tau_2).
\end{cases}
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We assume that the following assumptions are fulfilled:

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H.II: $EI(x)$ *and* $\rho(x)$ *are in* $C^4[0, \ell]$ *and there exists two positive constants* ρ_0 *and* EI_0 *such that*

$$0 < \rho_0 \leq \rho(x), \ \ 0 < EI_0 \leq EI(x), \quad \forall x \in [0, \ell].$$

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H.IV: *The actuator* r *obeys the conditions:*

Well-posedness of the Problem

All eigenvalues of the matrix A are in the open left-half plane and the triplet (A, b, c) is both observable and controllable.
The actuator transfer function $G(s) = \alpha + c^\top (sI - A)^{-1} b$ is a strictly positive real function in the sense that there exists a constant $\sigma > 0$ such that $\alpha > \sigma$ and $\Re\{G(i\varrho)\} > \sigma$, for any $\varrho \in \mathbb{R}$.

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We deduce that for each $n \times n$ symmetric positive definite matrix Q, there exist an $n \times n$ symmetric positive definite matrix P, a constant vector $q \in \mathbb{R}^n$ and $\nu > 0$ sufficiently small such that:

$$A^\top P + PA = -qq^\top - \nu Q, \quad (4)$$

$$Pb - \frac{c}{2} = \sqrt{\alpha - \sigma}\, q. \quad (5)$$

Well-posedness of the Problem

Then, let

$$H_c^n = \left\{ f \in H^n(0,1); f(0) = f_x(0) = 0 \right\}, \quad \text{for} \quad n = 2, 3, \cdots$$

Consider the state space $\mathcal{H}$ defined by

$$\mathcal{H} = H_c^2 \times L^2(0,\ell) \times L^2\Big((0,1) \times (\tau_1, \tau_2)\Big) \times \mathbb{R}^n \times \mathbb{R} := \mathcal{U} \times \mathbb{R},$$

equipped with the real inner products candidates:

$$\langle (y,z,u,r,\omega), (\tilde{y},\tilde{z},\tilde{u},\tilde{r},\tilde{\omega}) \rangle_{\mathcal{H}} = \langle (y,z,u,r), (\tilde{y},\tilde{z},\tilde{u},\tilde{r} \rangle_{\mathcal{U}} + \omega\tilde{\omega}$$

$$\langle (y,z,u,r), (\tilde{y},\tilde{z},\tilde{u},\tilde{r} \rangle_{\mathcal{U}} = \int_0^\ell \left(EI(x) y_{xx} \tilde{y}_{xx} - \varpi^2 \rho(x) y \tilde{y} + \rho(x) z \tilde{z} \right) dx$$

$$+ |\beta| \int_{\tau_1}^{\tau_2} s\lambda(s) \left\{ \int_0^1 u(\zeta,s) \tilde{u}(\zeta,s) d\zeta \right\} ds + 2\tilde{r}^\top P r.$$

Well-posedness of the Problem

The system can be written in $\mathcal{H}$ as a Cauchy problem

$$\begin{cases} \Phi_t(t) = \left[\begin{pmatrix} \mathcal{A} & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{B} \right] \Phi(t), \\ \Phi(0) = \Phi_0 = (y_0, y_1, f_0, r_0, \omega_0), \end{cases} \tag{6}$$

where $z = y_t$, $\Phi = (y, z, u, r, \omega)$, and $\mathcal{A}$ is a linear operator

$$\mathcal{D}(\mathcal{A}) = \Big\{ (y, z, u, r) \in H_c^4 \times H_c^2 \times L^2\Big((\tau_1, \tau_2); H^1(0, \ell) \Big) \times \mathbb{R}; \\ u(0, \cdot) = z(\ell),\ y_{xx}(\ell) = 0, \\ (EI(x) y_{xx})_x(\ell) = c^\top r + \alpha z(\ell) + \beta \int_{\tau_1}^{\tau_2} \lambda(s) u(1, s)\ ds \Big\}$$

$$\mathcal{A}(y, z, u, r) = \Big(z, -\frac{1}{\rho(x)} \big(EI(x) y_{xx} \big)_{xx} + \varpi^2 y, -s^{-1} u_\zeta, Ar + bu(0, \cdot) \Big). \tag{7}$$

Well-posedness of the Problem

$$\mathcal{B}\Phi = \left(0, (\omega^2 - \varpi^2)y, 0, 0, \frac{-\gamma\,(\omega - \varpi) - 2\,\omega < \rho(x)y, z >_{L^2(0,\ell)}}{I_d + \|\sqrt{\rho(x)}\,y\|^2_{L^2(0,\ell)}}\right).$$

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We have:

Theorem 1

Under the assumptions **H.I-H.VI** *and the condition*

$$|\beta| \int_{\tau_1}^{\tau_2} \lambda(s)\; ds \le \sigma < \alpha, \tag{8}$$

the operator $\mathcal{A}$ defined by (7) generates a C_0-semigroup of contractions $S(t)$ on $\mathcal{U}$.

Well-posedness of the Problem

Proof.

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- Integration by parts gives

$$\begin{aligned} \langle \mathcal{A}\phi, \phi \rangle_{\mathcal{H}} \leq & -\left(\sigma - |\beta| \int_{\tau_1}^{\tau_2} \lambda(s)\, ds \right) z^2(\ell) \\ & - \left(\sqrt{\alpha - \sigma} z(\ell) - r^\top q \right)^2 - \nu r^\top Q r. \end{aligned} \tag{9}$$

for any $\phi = (y, z, u, r) \in \mathcal{D}(\mathcal{A})$.

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for any $\phi = (y, z, u, r) \in \mathcal{D}(\mathcal{A})$.

- $\mathrm{R}(\lambda I - \mathcal{A}) = \mathcal{U}$, for $\lambda > 0$.

□

Well-posedness of the Problem

Theorem 2

Assume that the assumptions **H.I-H.IV** *hold and the feedback gains of the force control satisfy the condition (8). Then, for any initial condition* $\Phi_0 \in \mathcal{D}(\mathcal{A}) \times \mathbb{R}$*, the system (6) possesses a unique classical global bounded solution* $\Phi(t) \in \mathcal{D}(\mathcal{A}) \times \mathbb{R}$*. However, if* $\Phi_0 \in \mathcal{H}$ *then the system (6) has a unique mild global bounded solution* $\Phi(t) \in \mathcal{H}$*.*

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Proof.

- $\mathcal{A}$ generates a C_0-semigroup.
- $\mathcal{B}$ is differentiable on $\mathcal{H}$.

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Proof.

- $\mathcal{A}$ generates a C_0-semigroup.
- $\mathcal{B}$ is differentiable on $\mathcal{H}$.
- This leads to a local solution.

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- Propose the following Lyapunov functional:

$$
\begin{aligned}
V(t) &= \frac{I_d}{2}(\omega-\varpi)^2 + \frac{1}{2}(\omega-\varpi)^2\int_0^\ell \rho(x)y^2dx \\
&+ \frac{1}{2}\int_0^\ell (\rho(x)y_t^2 + EI(x)y_{xx}^2 - \varpi^2\rho(x)y^2)dx \\
&+ \frac{|\beta|}{2}\int_{\tau_1}^{\tau_2} s\lambda(s)\left\{\int_0^1 y_t^2(\ell,t-\zeta s)d\zeta\right\}ds \\
&+ r^\top Qr. \qquad (10)
\end{aligned}
$$

□

Stability of an Uncoupled System

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We first establish the uniform exponential stability in $\mathcal{U}$ of the semigroup $S(t)$ by means of the resolvent method:

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Proof.

It suffices to show that

$(p1)$ $\{i\xi;\, \xi \in \mathbb{R}\} \subset \rho(\mathcal{A})$;

$(p2)$ $\sup\{\|(i\delta - \mathcal{A})^{-1}\|_{\mathcal{U}};\, \delta \in \mathbb{R}\} < \infty$.

Stability of an Uncoupled System

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Stability of an Uncoupled System

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- $(p1)_a$ Ker $(i\xi I - \mathcal{A}) = \{0\}$;
  $(p1)_b$ $\mathrm{R}(i\xi I - \mathcal{A}) = \mathcal{U}$ for all real number $\xi \neq 0$.
- Proof of $(p2)$ by contradiction. This implies that there exists a sequence of real numbers $\delta_n \to \infty$ and a sequence of elements $\phi_n = (y^n, z^n, u^n, r^n) \in \mathcal{D}(\mathcal{A})$ satisfying

$$\|\phi_n\|_{\mathcal{U}} = \|y^n\|_{H_c^2} + \|z^n\|_{L^2(0,\ell)} + \|u^n\|_{L^2((0,1)\times(\tau_1,\tau_2))} + |r^n|_{\mathbb{C}^n} = 1, \tag{11}$$

and

$$\|(i\delta_n I - \mathcal{A})\phi_n\|_{\mathcal{U}} \longrightarrow 0, \quad \text{as } n \to \infty. \tag{12}$$

Stability of an Uncoupled System

Proof.

Equivalently, we have

$$i\delta_n y^n - z^n \equiv F^n \longrightarrow 0 \quad \text{in } H_c^2, \tag{13}$$

$$i\delta_n \rho(x) z^n + (EI(x) y_{xx}^n)_{xx} - \varpi^2 \rho(x) y^n \equiv G^n \longrightarrow 0 \text{ in } L^2(0,\ell), \tag{14}$$

$$i\delta_n s u^n + u_\zeta^n \equiv V^n \longrightarrow 0 \quad \text{in } L^2((0,1)\times(\tau_1,\tau_2)), \tag{15}$$

$$(i\delta_n I - A_1) r^n - b z^n(\ell) \equiv W^n \longrightarrow 0 \quad \text{in } \mathbb{C}^n, \tag{16}$$

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$$y^n(0) = y_x^n(0) = y_{xx}^n(\ell) = 0, \tag{17}$$

$$(EI(x) y_{xx}^n)_x(\ell) = c^\top r^n + \alpha z^n(\ell) + \beta \int_{\tau_1}^{\tau_2} \lambda(s) u^n(1,s)\, ds, \tag{18}$$

$$z^n(\ell) = u^n(0,s). \tag{19}$$

Stability of an Uncoupled System

Proof.

$$\begin{aligned} \|(i\delta_n I - \mathcal{A})\phi_n\|_{\mathcal{U}} &\geq \nu\,|(r^n)^* Q r^n| + \left|\sqrt{\alpha-\sigma} z^n(\ell) - (r^n)^* q\right|^2 \\ &+ \left(\sigma - |\beta| \int_{\tau_1}^{\tau_2} \lambda(s)\ ds\right) |z^n(\ell)|^2 , \end{aligned} \tag{20}$$

where the sign $*$ denotes the conjugate transpose.

Stability of an Uncoupled System

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$$\|(i\delta_n I - \mathcal{A})\phi_n\|_{\mathcal{U}} \geq \nu\,|(r^n)^* Q r^n| + \left|\sqrt{\alpha-\sigma} z^n(\ell) - (r^n)^* q\right|^2 + \left(\sigma - |\beta| \int_{\tau_1}^{\tau_2} \lambda(s)\, ds\right) |z^n(\ell)|^2, \qquad (20)$$

where the sign $*$ denotes the conjugate transpose.
This, together with (8)-(9) and (12), implies that, as $n \to \infty$

$$z^n(\ell) = u^n(0,s) \longrightarrow 0 \quad \text{in } \mathbb{C}, \qquad (21)$$

$$r^n \longrightarrow 0 \quad \text{in } \mathbb{C}^n. \qquad (22)$$

Stability of an Uncoupled System

Proof.

Combining (21) with (13), we have: $\delta_n y^n(\ell) \longrightarrow 0 \quad \text{in } \mathbb{C}.$

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Multiplying (15) by $\lambda(s)e^{-\delta_n(1-\zeta)}$ and then integrating with respect to ζ and s, we have

$$\int_{\tau_1}^{\tau_2} \lambda(s)u^n(1,s)\,ds \longrightarrow 0 \text{ in } \mathbb{C}, \tag{23}$$

and hence

$$(EI(x)y^n_{xx})_x(\ell) \longrightarrow 0 \quad \text{in } \mathbb{C}, \quad \text{as } n \to \infty. \tag{24}$$

Stability of an Uncoupled System

Proof.

Amalgamating (13) and (14), we get

$$(EI(x)y^n_{xx})_{xx} - \rho(x)(\varpi^2 + \delta_n^2)\, y^n = i\delta_n \rho(x) F^n + G^n. \tag{25}$$

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Let

$$a(x) = \int_x^\ell (\rho(s)/EI(s))^{1/4}\, ds, \tag{26}$$

and $\chi_n = \sqrt[4]{\delta_n^2 + \varpi^2}$. Next, multiply (25) by $\frac{1}{\chi_n} e^{-\chi_n a(x)}$, integrate by parts and use interpolation inequality, we obtain

$$\chi_n\, y^n_x(\ell) = (\delta_n^2 + \varpi^2)^{\frac{1}{4}}\, y^n_x(\ell) \longrightarrow 0 \text{ in } \mathbb{C}, \quad \text{as } n \to \infty. \tag{27}$$

Stability of an Uncoupled System

Proof.

Define

$$k(x) = e^{k_0 x} - 1, \quad \text{where} \quad k_0 = \max\left\{ \frac{\|\rho'\|_\infty}{\rho_0}, \frac{\|EI'\|_\infty}{EI_0} \right\}. \qquad (28)$$

Then, multiply (25) by $k(x)\, y_x^n$. Arguing as before, we have

$$\int_0^\ell \Big(3EI(x)k'(x) - k(x)EI'(x) \Big) |y_{xx}^n|^2 \, dx + \int_0^\ell \Big(k(x)\rho(x) \Big)' |\sqrt{\delta_n^2 + \varpi^2}\, y^n|^2 dx \longrightarrow 0. \qquad (29)$$

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Proof.

Since $3EI(x)k'(x) - k(x)EI'(x)$ and $\Big(k(x)\rho(x)\Big)'$ are positive functions in view of (28), it follows from (29) that

$$\|y^n\|_{H^2_c} \longrightarrow 0, \text{ and } \|z^n\|_{L^2(0,\ell)} = \|\delta_n y^n\|_{L^2(0,\ell)} \longrightarrow 0. \tag{30}$$

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Proof.

Since $3EI(x)k'(x) - k(x)EI'(x)$ and $\Big(k(x)\rho(x)\Big)'$ are positive functions in view of (28), it follows from (29) that

$$\|y^n\|_{H_c^2} \longrightarrow 0, \text{ and } \|z^n\|_{L^2(0,\ell)} = \|\delta_n y^n\|_{L^2(0,\ell)} \longrightarrow 0. \tag{30}$$

Lastly, going back to (15), we obtain
$\|u^n\|_{L^2((0,1)\times(\tau_1,\tau_2))} \to 0$, as $n \to \infty$. This together with (22) and (30) implies that as $n \longrightarrow \infty$
$\|\phi_n\|_{\mathcal{U}} = \|y^n\|_{H_c^2} + \|z^n\|_{L^2(0,\ell)} + \|u^n\|_{L^2((0,1)\times(\tau_1,\tau_2))} + |r^n|_{\mathbb{C}^n} \to 0,$
which contradicts (11) and hence $(p2)$ must hold. □

Exponential Stability for the Global System

The main result is:

Theorem 4

Suppose that the assumptions **H.I-H.III** *hold and*

$$|\beta| \int_{\tau_1}^{\tau_2} \lambda(s)\, ds < \sigma < \alpha. \tag{31}$$

Then, for any initial data $\Phi_0 \in \mathcal{D}(\mathcal{A}) \times \mathbb{R}$, *the corresponding solution* Φ *of the closed-loop system (6) exponentially tends to the equilibrium state* $(0_{\mathcal{U}}, \varpi)$ *in* $\mathcal{H}$ *as* $t \to \infty$.

Exponential Stability for the Global System

Proof.

Write the solution $\Phi(t)$ of the global system (6) stemmed from $\Phi_0 = (\phi_0, \omega_0) \in \mathcal{D}(\mathcal{A})$ as follows:

Exponential Stability for the Global System

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Write the solution $\Phi(t)$ of the global system (6) stemmed from $\Phi_0 = (\phi_0, \omega_0) \in \mathcal{D}(\mathcal{A})$ as follows:

$$\Phi(t) = \left(\tilde{\phi}(t), \omega(t)\right),$$

where $\tilde{\phi}(t) = (y, y_t, u, r)$ is the unique solution of the subsystem

$$\tilde{\phi}_t(t) = \mathcal{A}\tilde{\phi}(t) + (\omega^2(t) - \varpi^2)\mathcal{P}\tilde{\phi}(t), \tag{32}$$

where $\mathcal{P}$ is the bounded operator defined by

$$\mathcal{P}(f, g, h, v) = (0, f, 0, 0), \quad \text{for any } (f, g, h, v) \in \mathcal{U}.$$

Exponential Stability for the Global System

Proof.

In turn, $\omega(t)$ is the solution of the ODE

$$\frac{d}{dt}\omega(t) = \frac{-\gamma\,(\omega - \varpi) - 2\,\omega < \rho(x)y, y_t >_{L^2(0,\ell)}}{I_d + \|\sqrt{\rho(x)}\,y\|^2_{L^2(0,\ell)}}, \qquad (33)$$

- Use the variation of constants formula to (32).

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$\Longrightarrow$ Exponential stability of the solutions $\tilde{\phi}$ of (32).

- Use (33) to get the exponential stability of $\omega - \varpi$ in $\mathbb{R}$.

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This outcome is confirmed in Figure 3 for the angular velocity $\omega(t)$ which converges to the desired ϖ except in d).

Numerical simulations

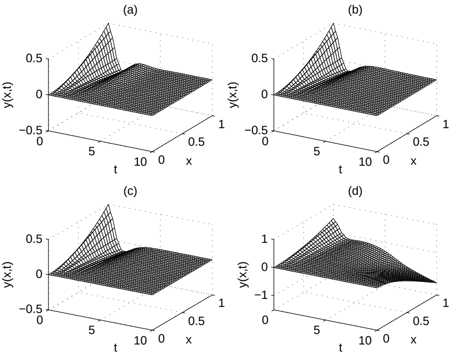

Fig. 2: Time evolution of the displacement $y(x,t)$. a) ϖ=2; b) ϖ=2.5; c) ϖ=2.6; d) ϖ=3.

Numerical simulations

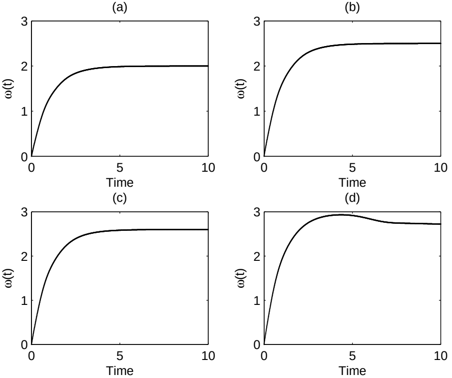

Fig. 3: Time evolution of the angular velocity $w(t)$. (a) ϖ=2; (b) ϖ=2.5; (c) ϖ=2.6; (d) ϖ=3.

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This observation is also clear from Figure 5 which depicts the time evolution of the solution of $y(x, t)$ for different values of β. These results are in line with the findings of Theorem 4.

Numerical simulations

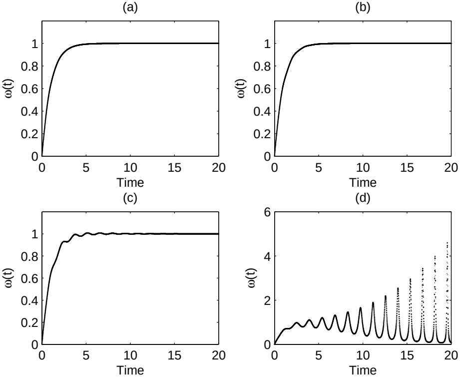

Fig. 4: Time evolution of the angular velocity $w(t)$. (a) $\beta=1$; (b) $\beta=5$; (c) $\beta=10$; (d) $\beta=15$.

Numerical simulations

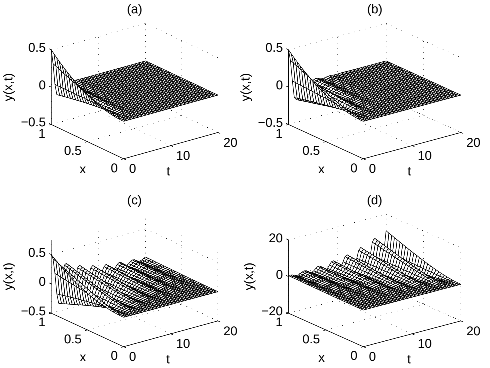

Fig. 5: Time evolution of the displacement $y(x,t)$. (a) β=1; (b) β=5; (c) β=10; (d) β=15.

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