

# Aggregation equation and collapse to singular measure

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Trends in PDE's and Related Fields

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# Collective behaviour and self-organization

## 1 Ant Trails



## 2 Bird flocks



## 3 fish schools



## Questions

- 1 Why a large number of individuals want to move through a flock ?
- 2 How large-scale complexity emerge from microscopic local rules without centralized coordination ?
- 3 Identify the mechanisms governing the interactions between individuals and their environment.

## Models

- Ideas from statistical physics( particle description),Kinetic theory (mean field limit).
- Hydrodynamic approach.

- Consider  $N$  particles moving through classical mechanics law :

$$\begin{cases} \dot{X}_i(t) = -\frac{1}{N} \sum_{j \neq i} \nabla K(X_i(t) - X_j(t)) \\ X_i(0) = X_i^0, i \in \{1, 2, \dots, N\} \end{cases}$$

We define the empirical measure

$$\mu_N(t) = \frac{1}{N} \sum_{i=1}^N \delta_{X_i(t)}$$

- If  $(\mu_N(0))_N$  converges weakly to  $\rho_0$ , then "formally"  $(\mu_N(t))_N$  converges weakly to the density  $\rho$  satisfying the continuity equation

$$\begin{cases} \partial_t \rho + \operatorname{div}(v\rho) = 0, & x \in \mathbb{R}^d, t \geq 0 \\ v = -\nabla K \star \rho, \\ \rho|_{t=0} = \rho_0. \end{cases}$$

# Aggregation equation with Newtonian potential

$$\begin{cases} \partial_t \rho + \operatorname{div}(v\rho) = 0, & x \in \mathbb{R}^2, t \geq 0 \\ v = -\nabla K \star \rho, & K(x) = \frac{1}{2\pi} \log|x| \\ \rho|_{t=0} = \rho_0. \end{cases}$$

- The vector-field  $v$  is compressible :

$$\operatorname{div} v = -\rho$$

- This equation is the compressible version of [Euler equations](#) where

$$v = \nabla^\perp K \star \rho = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \rho(y) dy$$

and  $\rho$  is the vorticity  $\omega = \partial_1 v^2 - \partial_2 v^1$ , which satisfies the transport equation

$$\partial_t \omega + v \cdot \nabla \omega = 0$$

## Classical solutions

- Wölibner (1933) : Euler equation is globally well-posed for  $\omega_0 \in C^\alpha, \alpha > 0$ . This follows from

$$\|\omega(t)\|_{L^\infty} = \|\omega_0\|_{L^\infty}$$

- Nieto-Poupaud-Soler; Bertozzi-Laurent-Léger : The aggregation equation is locally well-posed for  $\rho_0 \in C^\alpha, \alpha > 0 : \rho \in L^\infty([0, T^*), C^\alpha)$ .
- What about  $T^*$  ?

## Blow up

- The aggregation equation takes the form

$$\partial_t \rho + v \cdot \nabla \rho = \rho^2$$

- Using the characteristics we get

$$\rho(t, \psi(t, x)) = \frac{\rho_0(x)}{1 - t\rho_0(x)}$$

where  $\psi$  denotes the trajectories map :

$$\psi(t, x) = x + \int_0^t v(\tau, \psi(\tau, x)) d\tau$$

**Consequences** : Let  $\rho_0 \in C_c^\alpha, \alpha > 0$ , then

- Global existence is satisfied when  $\sup_{\mathbb{R}} \rho_0(x) \leq 0$ .
- Blow up in finite time happens if and only if  $\sup_{\mathbb{R}} \rho_0(x) > 0$ ,

$$T_\star = \frac{1}{\sup_{\mathbb{R}} \rho_0(x)} \quad \text{and} \quad \|\rho(t)\|_{L^\infty} = \frac{1}{T_\star - t}$$

## Yudovich solutions type

- **Yudovich** : Euler admits global unique solution  $\omega \in L^\infty(\mathbb{R}_+; L^1 \cap L^\infty)$
- **N-P-S, B-L-L** : Existence of solutions when  $\rho_0 \in L^1 \cap L^\infty$ , with

$$\rho \in L^\infty([0, T], L^\infty \cap L^1)$$

- For  $\rho_0 \in L^1 \cap C_c$

$$\|\rho(t)\|_{L^1} = \|\rho_0\|_{L^1}$$

- The blow up does not occur for  $L^1$  !



# Concentration phenomenon

- **Question** : The  $L^1$  norm is conserved but what about **concentration** phenomenon when  $t \rightarrow T^*$  ?

Two examples ( from B-L-L )

- 1 **Delta mass concentration** : Let  $\rho_0 = \frac{1}{\pi} \mathbf{1}_{D(0,1)}$  then weakly

$$\lim_{t \rightarrow 2\pi} \rho(t) dx \rightarrow \pi \delta_0$$

- 2 **Vortex sheet concentration** : Let  $\rho_0 = (\pi ab)^{-1} \mathbf{1}_{\mathcal{E}}$  with  $\mathcal{E}$  the ellipse centered at zero and with semi axes  $a$  and  $b$ . Then  $\rho(t) = (\pi a(t)b(t))^{-1} \mathbf{1}_{\mathcal{E}_t}$ , with  $\mathcal{E}_t$  an ellipse and it converges weakly to the **semi-cricle law**

$$\lim_{t \rightarrow \pi ab} \rho(t) dx = \frac{2\sqrt{x_0^2 - x_1^2}}{\pi x_0^2} \mathbf{1}_{[-x_0, x_0]} dx_1, \quad x_0 = a - b$$

## Time rescaling and vortex patch problem

- Let  $\rho_0 = \mathbf{1}_{D_0}$  with  $D_0$  a bounded domain, then

$$\rho(t, x) = \frac{\mathbf{1}_{\widehat{D}_t}}{1-t}, 0 \leq t < 1 = T_*, \quad \widehat{D}_t = \psi(t, D_0)$$

Notice that

$$|\widehat{D}_t| = (1-t)|D_0|$$

- Let  $\tau = -\ln(1-t)$  and  $D_\tau = \widehat{D}_{1-e^{-\tau}}$  then  $\mathbf{1}_{D_\tau}$  is a solution of

$$\partial_\tau \rho + v \cdot \nabla \rho = 0, \quad \rho_0 = \mathbf{1}_{D_0}, \quad v(\tau) = \nabla K \star \mathbf{1}_{D_\tau}, \quad \tau \in [0, +\infty),$$

One also has

$$\forall \tau \geq 0, \quad |D_\tau| = e^{-\tau}|D_0|$$

- Chemin** (Euler equation) : If  $\partial D_0 \in C^{1+\varepsilon}$  then  $\forall t \geq 0, \quad \partial D_t \in C^{1+\varepsilon}$ ,
- Bertozi-Garnett-Laurent-Verdera** (Aggregation) : similar result.

## Asymptotic behavior

- Recall that  $|D_t| = e^{-t}|D_0|$ .
- This implies that

$$\int_0^{+\infty} \|v(t)\|_{L^\infty} dt < \infty$$

- This allows to define the limit shape

$$D_\infty := \left\{ \lim_{t \rightarrow +\infty} \psi(t, X), X \in D_0 \right\}$$

- What about the geometric structure of  $D_\infty$  ?
- What about the weak limit of the probability measure :

$$dP_t = e^t \frac{\mathbf{1}_{D_t}}{|D_0|} dX, \quad X = (x, y) \in \mathbb{R}^2$$

- Numerical simulations (B-L-L) : Collapse to skeleton shapes.

## Concentration along skeleton structure

### Proposition

Let  $D_0$  be a simply connected domain symmetric with respect to an axis  $\Delta$ . Denote by  $L = \text{Length}(D_0 \cap \Delta)$ . There exists an absolute constant  $C$  such that if

$$L > C|D_0|^{\frac{1}{2}}$$

then the shape  $D_\infty$  contains a segment of size  $L - C|D_0|^{\frac{1}{2}}$ .

Thin initial domains along their axis of symmetry generate concentration to segments.

- Applications to **polygonal shapes**.

# One fold symmetric patches

**Main goal** : Find initial patches for which we can analyze the concentration to a collection of segments lying in the same straight line.

- Let  $D_0$  be a planar set symmetric with respect to the real axis with

$$D_0 = \{(x, y), x \in \mathbb{R}, -f(x) \leq y \leq f(x)\}, \quad f \in C_c^1(\mathbb{R}, \mathbb{R}_+)$$

- $D_0$  is defined by a graph and what about  $D_t$  ?
- Note that  $D_t$  is symmetric with respect to the real axis for any  $t \geq 0$ .

## Strategy

- Write the graph equation.
- Study local well-posedness : Hölder and continuous Dini spaces.
- Global well-posedness with small initial data.
- Collapse to singular measure.

## Graph reformulation

- Let  $s \mapsto \gamma_t(s)$  be any parametrization of the boundary  $\partial D_t$ , then

$$(\partial_t \gamma_t(s) - v(t, \gamma_t(s))) \cdot \vec{n}(\gamma_t(s)) = 0$$

with  $\vec{n}(\gamma_t)$  being a normal vector to the boundary at the point  $\gamma_t(s)$

- By taking the parametrization  $x \mapsto (x, f(t, x))$  we get

$$\begin{cases} \partial_t f(t, x) + u_1(t, x) \partial_x f(t, x) = u_2(t, x), & t \geq 0, x \in \mathbb{R} \\ f(0, x) = f_0(x). \end{cases}$$

where  $(u_1, u_2)(t, x)$  is the trace of  $(v_1, v_2)$  at the point  $X = (x, f(t, x))$ .

- One has the expressions

$$u_1(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} \left\{ \arctan \left( \frac{f_t(x+y) - f_t(x)}{y} \right) + \arctan \left( \frac{f_t(x+y) + f_t(x)}{y} \right) \right\} dy$$

$$u_2(t, x) = \frac{1}{4\pi} \int_{\mathbb{R}} \log \left[ \frac{y^2 + (f_t(x+y) - f_t(x))^2}{y^2 + (f_t(x+y) + f_t(x))^2} \right] dy.$$

## Qualitative properties

- Maximum principle :

$$\forall t \geq 0, \quad x \in \mathbb{R}, \quad 0 \leq f(t, x) \leq \|f_0\|_{L^\infty}.$$

This can be derived from the fact that

$$u_2(t, x) \leq 0.$$

- Confinement of the support : If  $\text{supp} f_0 \subset [a, b]$  then

$$t \geq 0, \quad \text{supp} f(t) \subset [a, b]$$



## Functional setting

- Hölder and Dini spaces : let  $s \in (0, 1)$  denote by  $C^s$  the usual Hölder space and

$$f \in C_D \iff \|f\|_{L^\infty} + \int_0^1 \frac{\mu(r)}{r} dr < \infty$$

$$\text{with } \mu(r) = \sup_{|x-y| \leq r} |f(x) - f(y)|$$

- Dini space is an algebra, contrary to Besov space  $B_{\infty,1}^0$ .
- Beirao da Veiga (1984) : Euler equations are global when the vorticity belongs to Dini space. This is a consequence of the composition law :

$$\|f \circ \psi\|_{C_D} \leq C \|f\|_{C_D} \log(1 + \|\nabla \psi\|_{L^\infty})$$

- We have the embedding : for any  $s \in (0, 1)$

$$C^s \hookrightarrow C_D \hookrightarrow B_{\infty,1}^0 \hookrightarrow C_b$$

Local/global well-posedness

Denote by  $X = C^s, s \in (0, 1), C_D$ .

## Theorem (H.-Li (2018))

Let  $f_0$  be a positive compactly supported function, s.t.  $f_0' \in X$ . Then

- 1 The graph equation admits a unique local solution s.t.  $f' \in L^\infty([0, T], X)$ .
- 2 There exists  $\varepsilon > 0$  such that if

$$\|f_0'\|_{C^s} \leq \varepsilon,$$

then  $T = +\infty$ . Moreover

$$\|f(t)\|_{L^\infty} + \|f'(t)\|_{L^\infty} \leq C_0 e^{-t}$$

## Theorem (H.-Li (2018))

Let  $f_0$  be a positive compactly supported function, s.t.  $f_0' \in C^s$  with *small norm*. Assume that  $\text{supp } f_0$  is the union of  $n$ -disjoint segments.

- ① There exists a compact set  $D_\infty \subset \mathbb{R}$  composed of exactly  $n$ -disjoint segments,

$$\forall t \geq 0, \quad d_H(D_t, D_\infty) \leq C_0 e^{-t}$$

- ② The probability measure  $dP_t = e^t \frac{1_{D_t}}{|D_0|} dX$  converges weakly as  $t \rightarrow +\infty$  to the probability measure

$$dP_\infty(X) = \Phi(x) \mathbf{1}_{D_\infty} dx, \quad X = (x, y)$$

with  $\Phi \in C^\alpha, \forall \alpha \in (0, 1)$  and

$$\begin{aligned} \Phi(x) &= 2 \lim_{t \rightarrow +\infty} e^t f(t, x) \\ &= \frac{f_0(\psi_\infty^{-1}(x))}{\|f_0\|_{L^1}} e^{g(x)} \end{aligned}$$

where  $\psi_\infty(x) = \lim_{t \rightarrow +\infty} \psi_1(t, x)$  and  $\psi_1$  is the flow associated to  $u_1$ . The function  $g$  is recovered through the full dynamics of the graph.

## Slope equation

- Set  $g(t, x) = \partial_x f(t, x) = f'(t, x)$  and define the operators

$$\Delta_y^\pm f(x) = f(x + y) \pm f(x)$$

then

$$\partial_t g + u_1 \partial_x g = F^-(t, x) - F^+(t, x)$$

with

$$F^\pm(t, x) = \frac{1}{2\pi} \text{p.v.} \int_{\mathbb{R}} \frac{[\Delta_y^\pm f(x) \pm y f'(x)] \Delta_y^\pm f'(x)}{y^2 + [\Delta_y^\pm f(x)]^2} dy$$

- For local well-posedness we need to estimate

$$\|F^\pm(t)\|_X, \quad X = C^s, C_D$$

## Estimate of $F^-$

- Recall that

$$F^-(t, x) = \frac{1}{2\pi} \text{p.v.} \int_{\mathbb{R}} \frac{[\Delta_y^- f(x) - yf'(x)] \Delta_y^- f'(x)}{y^2 + [\Delta_y^- f(x)]^2} dy$$

- It is connected to Cauchy operator associated to the graph  $f$  :

$$\mathcal{C}_f g(x) = \int_{\mathbb{R}} \frac{g(x+y) - g(x)}{y + i(f(x+y) - f(x))} dy$$

- [Coifman, McIntosh, Meyer](#) : If  $f \in W^{1,\infty}$  , then for  $p \in (1, \infty)$ ,  $\mathcal{C}_f : L^p \rightarrow L^p$  is continuous.
- [Wittmann](#) : If  $K$  is a compact,  $0 < s < 1$ ,  $f \in C^{1+s}$  then  $\mathcal{C}_f : C_K^s \rightarrow C^s$  is continuous.
- This operator "could" fail to be continuous on Dini space !

## Truncated bilinear Cauchy operators

- We shall be concerned with : for  $M > 0, \theta \in [0, 1]$

$$C_f^\theta(g, h)(x) = \int_{-M}^M \frac{[g(x + \theta y) - g(x)][h(x + y) - h(x)]}{y + i(f(x + y) - f(x))} dy$$

### Proposition

Let  $0 < s < 1, K$  be a compact and  $X = C_K^s, C_{D,K}$ . Then we have the estimates

$$\|C_f^\theta(g, h)\|_X \leq C [1 + \|f'\|_X^3] (\|g\|_D \|h\|_X + \|h\|_D \|g\|_X)$$

As an application :

$$\|F^-\|_X \leq C \|f'\|_D [\|f'\|_X + \|f'\|_X^3]$$

## Estimate of $F^+$

- Recall that

$$F^+(t, x) = \frac{1}{2\pi} \text{p.v.} \int_{\mathbb{R}} \frac{[\Delta_y^+ f(x) + yf'(x)] \Delta_y^+ f'(x)}{y^2 + [\Delta_y^+ f(x)]^2} dy$$

- It is connected to Cauchy operator of the second kind : let  $\alpha, \beta \in [0, 1]$

$$\mathcal{T}_f^{\alpha, \beta} g(x) = \text{p.v.} \int_{\mathbb{R}} \frac{y g(\alpha x + \beta y)}{y^2 + [f(x+y) + f(x)]^2} dy$$

## Proposition

Let  $f$  be **positive**, compactly supported in  $K$  and  $f' \in C_D$ . Then

- The operator  $\mathcal{T}_f^{\alpha, \beta} : C_{D, K} \rightarrow L^\infty$  is continuous independ. on  $\alpha, \beta$
- The modified operator  $f' \mathcal{T}_f^{\alpha, \beta} : C_{D, K} \rightarrow C_{D, K}$  is continuous. The continuity constant depends only on  $|\ln \beta|$  and finite for  $\beta = 0$ .
- Similar estimates in  $C_K^s$ .
- Application :

$$\|F^+\|_X \leq C \left(1 + \|f'\|_D^{\frac{1}{3}}\right) [\|f'\|_X + \|f'\|_X^{16}]$$

## One key estimate

### Lemma

Let  $K$  be a compact set of  $\mathbb{R}$  and  $f : \mathbb{R} \rightarrow \mathbb{R}_+$  be a continuous **positive** function supported in  $K$  such that  $f' \in C_D$ . Then we have,

$$\forall x \in \mathbb{R}, \quad |f'(x)| \leq C \frac{\|f'\|_{C_D} (1 + \ln_+(1/\|f'\|_D))}{1 + \ln_+(\frac{1}{f(x)})},$$

If in addition  $f' \in C^s(\mathbb{R})$  with  $s \in (0, 1)$ , then

$$\forall x \in \mathbb{R}, \quad |f'(x)| \leq C \|f'\|_s^{\frac{1}{1+s}} [f(x)]^{\frac{s}{1+s}}$$

and the constant  $C$  depends only on  $s$ .



## Weak damping and global existence

- Here  $X = C_K^s, 0 < s < 1$ . Recall that

$$\|F^-\|_{C^s} \leq C \|f'\|_D [\|f'\|_{C^s} + \|f'\|_{C^s}^3]$$

and

$$\|F^+\|_{C^s} \leq C \left(1 + \|f'\|_D^{\frac{1}{3}}\right) [\|f'\|_{C^s} + \|f'\|_{C^s}^{16}]$$

- By interpolation

$$\begin{aligned} \|f'(t)\|_D &\lesssim \|f(t)\|_{L^1}^\theta \|f'(t)\|_{C^s}^{1-\theta} \\ &\lesssim e^{-Ct} \|f'(t)\|_{C^s}^{1-\theta} \end{aligned}$$

- We establish the refined estimate :

$$F^+(t, x) = f'(x) + L(t, x) + N(t, x)$$

$$\|L(t)\|_{C^s} \leq \|f'\|_{C^s} + C \|f'\|_D^5 \|f'\|_{C^s}$$

$$\|N(t)\|_{C^s} \leq C \|f'\|_D^{\frac{1}{3}} [\|f'\|_{C^s} + \|f'\|_{C^s}^{16}]$$

- $g = f'$  satisfies

$$\partial_t g + u_1 \partial_x g + g = -L(t, x) - N(t, x) + F^-(t, x).$$

- Define the probability measure

$$dP_t = e^t \mathbf{1}_{D_t} dX, \quad X = (x, y), |D_0| = 1$$

Note that  $\text{supp } dP_t \subset K_0$  a fixed compact independent of the time

- Let  $\varphi \in C_c^\infty(\mathbb{R}^2)$ , then by Fubini

$$\begin{aligned} I_t &:= \int_{\mathbb{R}^2} \varphi(x, y) dP_t(X) \\ &= e^t \int_{\mathbb{R}} \int_{-f(t,x)}^{f(t,x)} \varphi(x, y) dy dx \end{aligned}$$

- Taylor expansion on  $y$  :  $\varphi(x, y) = \varphi(x, 0) + y\eta(x, y)$ ,  $\eta \in L^\infty(\mathbb{R}^2)$  and

$$I_t = 2e^t \int_{\mathbb{R}} f(t, x) \varphi(x, 0) dx + O(e^{-t}) \quad \text{since } \|f(t)\|_{L^\infty} \leq Ce^{-t}$$

- Main result** : there exists  $\Phi \in C^\alpha(\mathbb{R})$ ,  $\forall \alpha \in (0, 1)$  such that

$$\|e^t f(t) - \Phi\|_{L^\infty} = O(e^{-\epsilon t})$$

One has

$$\partial_t f(t, x) + u_1 \partial_x f(t, x) + f(t, x) = -f(t, x)R(t, x) \quad (1)$$

with

$$\begin{aligned} \|R(t)\|_{L^\infty} &\leq \|f'(t)\|_D (1 + \|f'(t)\|_\infty^5) \\ &\leq Ce^{-\eta t}, \quad \eta > 0. \end{aligned} \quad (2)$$

From the characteristic method we get the representation

$$e^t f(t, \psi_1(t, x)) = f_0(x) e^{\int_0^t R(\tau, \psi_1(\tau, x)) d\tau}. \quad (3)$$

Thus

$$\lim_{t \rightarrow +\infty} \|e^t f(t, \psi_1(t, \cdot)) - R_2(\cdot)\|_{L^\infty} = 0,$$

with

$$R_2 : x \mapsto f_0(x) e^{\int_0^{+\infty} R(\tau, \psi_1(\tau, x)) d\tau}. \quad (4)$$

Thank you !!